Constructive solutions to Pólya-Schur problems

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Abstract

We present constructive solutions to the following Pólya-Schur problems concerning linear operators on the space of univariate polynomials: Given subsets Ω_1 and Ω_2 of the complex plane, determine operators that map all polynomials having no zeros in Ω_1 to polynomials having no zeros in Ω_2, or to the zero polynomial. We describe an explicit class consisting of rank 1 operators and product-composition operators that solve the stated problems for arbitrary Ω_1 and Ω_2; and this class is shown to comprise *all* solutions when Ω_1 is bounded and Ω_2 has non-empty interior. The latter result encompasses a number of open problems and, moreover, gives explicit solutions in cases of circular domains $\Omega_1 = \Omega_2$ where existing characterizations are non-constructive. The paper also treats problems stemming from digital signal processing that are analogous to Pólya-Schur problems. Specifically, we describe all bounded linear operators on Hardy space that preserve the class of outer functions, as well as those that preserve shifted outer functions.

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1 Introduction

This paper presents constructive solutions valid for all Pólya-Schur problems, and proves that for a wide class of Pólya-Schur problems, the given constructive solutions comprise *all* solutions. The paper also treats functional analytic problems in the context of Hardy spaces that stem from digital signal processing. The main technical engine of the paper is Theorem 2 in Section 3.2, proved using analytic and functional analytic arguments, that allows both Pólya-Schur and Hardy space problems to be treated simultaneously.

The basic problem under consideration is to identify which linear operators acting on polynomials transform *stability* with respect to a given regions in the complex plane. Given a set $\Omega \subset \mathbb{C}$, a complex univariate polynomial $p \in \mathbb{C}[z]$ is said to be Ω-stable when p has no zeros in Ω. The set of all Ω-stable polynomials is denoted $\mathcal{P}(\Omega)$. The general univariate Pólya-Schur problem is:

Problem A Given arbitrary subsets $\Omega_1, \Omega_2 \subset \mathbb{C}$, determine all linear operators $A : \mathbb{C}[z] \to \mathbb{C}[z]$ such that

$$A(\mathcal{P}(\Omega_1)) \subset \mathcal{P}(\Omega_2) \cup \{0\}.$$

Problem A has a long history, and special cases have received considerable attention in recent years. The terminology harkens back almost a century to the work [14] of Pólya and Schur dealing with polynomials having real zeros, corresponding to the case $\Omega_1 = \Omega_2 = \mathbb{C} \setminus \mathbb{R}$. Pólya and Schur's characterization of multiplier sequences, which are diagonal operators with respect to the monomial basis for $\mathbb{C}[z]$, spawned a great deal of effort toward constructing concrete examples of operators conforming to their non-constructive characterization. Examples from the last thirty years include [7], [11] and [2], among many others; see [12, Chapter 8] for a summary of earlier developments. More recently, in [5], Borcea and Brändén obtained characterizations in the spirit of Pólya and Schur for circular domains, where the boundary of $\Omega_1 = \Omega_2$ is either a circle or a line, thereby greatly generalizing the seminal result on multiplier sequences. They also highlighted important cases of circular domains where no characterization is known, including the closed disk $\Omega_1 = \Omega_2 = \overline{\mathbb{D}}$. In a series of papers, [3], [4] and [6], Borcea and Brändén subsequently extended their characterizations to the multivariate setting and applied the extension to various problems including that considered by Lee and Yang in [16], as well as, with Liggett, to developing a theory of negative dependence. The survey [15] by Wagner gives an overview of these devel-

opments. As with Pólya and Schur's original work, Borcea and Brändén's characterization of stable preserving operators also opens up a potential line of research aimed at bridging their characterizations to explicit, constructive solutions.

The present paper concentrates on the univariate setting and treats Problem A with the objective of both characterizing solutions and constructing solutions explicitly. To that end, a simple, constructive sufficient condition is given for an operator to be a solution of Problem A. A major result of the paper consists in showing that this simple, constructive condition is also necessary under very general conditions, namely when Ω_1 is bounded and Ω_2 has non-empty interior. In particular, Problem A is given a complete, constructive solution in the previously open cases of the closed disk and the punctured disk, among many others. And in cases where a non-constructive characterization was previously known, such as $\Omega_1 = \Omega_2 = \mathbb{D}$, the present paper completes the bridge from the known characterization to explicit solutions.

These results show that the nature of the solution to Problem A when Ω_1 is bounded is not sensitive to geometric details such as circularity, or whether Ω_1 is open or closed. In light of the long history of Pólya-Schur problems, the solution itself is astonishingly simple to describe: either A is rank 1, mapping $\mathbb{C}[z]$ to the complex line through a fixed Ω_2-stable polynomial; or A is a product-composition operator consisting of (right) composition with a polynomial that maps Ω_2 into Ω_1, followed by multiplication with an Ω_2-stable polynomial.

The following functional analytic problem, stemming from digital signal processing, is also treated in the present paper. Let H^2 denote the Hardy space on the disk, consisting of all analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n : \mathbb{D} \to \mathbb{C},$$

having square-summable Taylor coefficients. Note that H^2 is a Hilbert space with respect to the norm $||f||^2 = \sum_{n=0}^{\infty} |a_n|^2$. A function $f \in H^2$ is said to be outer if H^2 is the closed linear span of the functions $z^n f(z)$ $(n \geq 0)$. And if f is outer, then a function of the form $z^n f(z)$, where $n \geq 0$, is said to be shifted outer. Let $\mathcal{O} \subset H^2$ denote the set of all outer functions, and let $\widetilde{\mathcal{O}} \subset H^2$ denote the set of all shifted outer functions.

Problem B Given $X = \mathcal{O}$ or $X = \widetilde{\mathcal{O}}$, determine all bounded linear operators $A : H^2 \to H^2$ such that $A(X) \subset X$.

The solution [8] of Problem B in the case $X = \widetilde{\mathcal{O}}$ is a very recent breakthrough that set the stage for vastly more general results. Indeed the present paper is an outgrowth of the realization by the first author that the functional analytic methods used for the case of shifted outer functions could be modified and extended to cover not only the case $X = \mathcal{O}$ but also Problem A when Ω_1 is bounded and Ω_2 has non-empty interior.

2 Background Material

For reference purposes, the present section compiles notation and a variety of needed results from complex analysis and complex functional analysis.

2.1 Notation

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk in the complex plane, and write $\mathbb{C}[z]$ for the set of all univariate polynomials having complex coefficients. Given a set $\Omega \subset \mathbb{C}$, the algebra of all analytic functions $f : \Omega \to \mathbb{C}$ is denoted by $\mathcal{A}(\Omega)$. Given sets $\Omega \subset \Omega' \subset \mathbb{C}$, a function $f : \Omega' \to \mathbb{C}$ is said to be Ω-stable if $f(z) \neq 0$ for every $z \in \Omega$. The set of all Ω-stable functions in $\mathcal{A}(\Omega')$ is denoted by $\mathcal{S}(\Omega)$. In most of what follows, Ω is an open connected subset of $\Omega' = \mathbb{D}$; in any case the larger set Ω' should be clear from context. The symbol $\mathcal{P}(\Omega)$ denotes the set of all Ω-stable polynomials.

Given a set $\Omega \subset \mathbb{C}$, the symbol $\mathcal{T}_2(\Omega)$ denotes the set of all linear operators $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ such that

$$A \text{ has rank at least 2, and } A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\}.$$

The letter $\mathcal{T}$ in the symbol $\mathcal{T}_2(\Omega)$ refers to the property that the operators *transform* $\mathbb{D}$-stability to Ω-stability, while the index 2 indicates the minimum rank.

2.2 Analytic and entire functions

Given a connected open set Ω, the vector space $\mathcal{A}(\Omega)$ is complete with respect to the topology of uniform convergence on compact sets, in which a given sequence $\{f_n\}$ converges to a function in f in $\mathcal{A}(\Omega)$ if and only if

$$\sup_{z \in K} |f_n(z) - f(z)| \to 0 \text{ as } n \to \infty \text{ for every compact set } K \subset \Omega.$$

This is the standard topology for analytic functions; in the present paper continuity of an operator mapping a topological space into $\mathcal{A}(\Omega)$ refers by

default to the topology of uniform convergence on compact sets, unless the contrary is explicitly stated. Hurwitz's Theorem [1, p. 178] asserts that with respect to this topology the closure of $\mathcal{S}(\Omega)$ in $\mathcal{A}(\Omega)$ is $\mathcal{S}(\Omega) \cup \{0\}$.

Concerning entire functions $f : \mathbb{C} \to \mathbb{C}$, Hadamard's Theorem [1, Theorem 8, p. 209] relates the rate of growth of $|f(z)|$, as $|z| \to \infty$, to the Weierstrass product form for f. For $r > 0$, let $M(r)$ denote the maximum value of $|f(z)|$ on the disk $|z| \le r$. The order of f is defined as

$$s = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

Let $\{z_n\}_{n=1}^N$ be the sequence of zeros of f, excluding 0, in increasing order

$$|z_n| \le |z_{n+1}|,$$

with each zero repeated according to its multiplicity. Either $N = \infty$ or N is a finite integer. In order to avoid having to make this distinction, the following formalism will be used: in the case where N is finite, set $z_{N+k} = \infty$ for each $k \ge 1$, so that the full sequence $\{z_n\}_{n=1}^\infty$ is always defined. With this formalism, Hadamard's Theorem states that if the order s of a function f is finite, then there are nonnegative integers $\gamma \le s$ (called the *genus* of f) and ν (the order of the origin as a zero of f) such that the Weierstrass product of f has the form

$$f(z) = e^{p(z)} z^\nu \prod_{n=1}^\infty e^{\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{\gamma}\left(\frac{z}{z_n}\right)^\gamma} \left(1 - \frac{z}{z_n}\right),$$

where p is a polynomial of degree at most γ. In the case $\gamma = 0$ this reduces to

$$f(z) = Cz^\nu \prod_{n=1}^\infty \left(1 - \frac{z}{z_n}\right).$$

2.3 Hardy space

As mentioned already, H^2 denotes the Hardy space on the disk, consisting of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ whose Taylor series

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

have square-summable coefficients. There is a well developed theory of Hardy space; the books [10] by Hoffman and [13] by Martínez-Avendaño

and Rosenthal are excellent references. The vector space H^2 is a Hilbert space with respect to the norm $||f||^2 = \sum_{n=0}^{\infty} |a_n|^2$, with the mapping

$$\sum_{n=0}^{\infty} a_n z^n \mapsto (a_0, a_1, a_2, \ldots)$$

an isometry from H^2 onto $l^2(\mathbb{Z}_+)$; here $\mathbb{Z}_+$ denotes the nonnegative integers, including 0. Every $f \in H^2$ has an extension (also denoted f) to the closed unit disk $\overline{\mathbb{D}}$, with values $f(e^{i\theta})$ being defined almost everywhere on the unit circle S^1. The norm of f is also given by its boundary function,

$$||f||^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta.$$

Note that

$$\mathbb{C}[z] \subset H^2 \subset \mathcal{A}(\mathbb{D})$$

is a sequence of proper inclusions, with $\mathbb{C}[z]$ dense in H^2 with respect to the norm topology and also dense in $\mathcal{A}(\mathbb{D})$ with respect to uniform convergence on compact sets. The norm topology on H^2 is strictly finer than the topology of uniform convergence on compact sets, so that the inclusion map

$$\iota : H^2 \to \mathcal{A}(\mathbb{D})$$

is continuous. Moreover, any continuous linear operator from $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})$ has a continuous restriction from $H^2 \to \mathcal{A}(\mathbb{D})$, where continuity of the restricted map is with respect to the norm topology on H^2, and the standard topology on $\mathcal{A}(\mathbb{D})$. Also, the original operator is completely determined by its restriction to H^2, since H^2 is dense in $\mathcal{A}(\mathbb{D})$. Additionally, any continuous linear operator from $H^2 \to H^2$ is automatically continuous when viewed as a map from $H^2 \to \mathcal{A}(\mathbb{D})$, by continuity of the inclusion map. The foregoing remarks show that the set of all continuous linear operators

$$A : H^2 \to \mathcal{A}(\mathbb{D})$$

includes all continuous linear operators on H^2 as well as all continuous linear operators on $\mathcal{A}(\mathbb{D})$.

A function $f \in H^2$ is *outer* if and only if $f(0) \neq 0$ and

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta.$$

Equivalently, f is outer if and only of the closed linear span of the set $\{z^n f(z) \,|, n \geq 0\}$ is all of H^2. In terms of the sequence $(a_0, a_1, a_2, \ldots)$ of

coefficients of f, f is outer if and only if $(a_0, a_1, a_2, \ldots)$ is a cyclic vector of the right shift operator

$$(x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, \ldots).$$

Let $\mathcal{O}$ denote the set of all outer functions in H^2. A shifted outer function is defined to be a function $g \in H^2$ of the form $g(z) = z^n f(z)$ for some $n \in \mathbb{Z}_+$, where $f \in \mathcal{O}$. Thus the coefficients of g are obtained from those of $f \in \mathcal{O}$ by n iterates of the right shift operator. The set of all shifted outer functions is denoted $\widetilde{\mathcal{O}}$. Letting $\mathbb{D}_*$ denote the punctured unit disk $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$, note that

$$\mathcal{P}(\mathbb{D}) \subset \mathcal{O} \subset \mathcal{S}(\mathbb{D}), \text{ and } \mathcal{P}(\mathbb{D}_*) \subset \widetilde{\mathcal{O}} \subset \mathcal{S}(\mathbb{D}_*).$$

In other words, all $\mathbb{D}$-stable polynomials are outer functions, and all outer functions are $\mathbb{D}$-stable; while all $\mathbb{D}_*$-stable polynomials are shifted outer, and all shifted outer functions are $\mathbb{D}_*$-stable. Each of the foregoing inclusions is proper.

3 Main Results: the Structure Theorem and its Implications

3.1 Multiplication and composition operators

Multiplication operators and composition operators are two classes of linear operators having particular importance in the context of Pólya-Schur problems; notation for these operators is fixed as follows. Given $\psi \in \mathcal{A}(\mathbb{D})$,

$$M_\psi : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$$

denotes the operation of multiplication by ψ,

$$(M_\psi p)(z) = \psi(z)p(z)$$

for every $p \in \mathbb{C}[z]$ and $z \in \mathbb{D}$. Given a function $\varphi \in \mathcal{A}(\mathbb{D})$,

$$C_\varphi : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$$

denotes the operation of (right) composition with φ,

$$(C_\varphi p)(z) = p \circ \varphi(z) = p(\varphi(z))$$

for every $p \in \mathbb{C}[z]$ and $z \in \mathbb{D}$. The same symbols M_ψ and C_φ will be used for the obvious extensions of these operators to other domains such as $\mathcal{A}(\mathbb{D})$ or the Hardy space H^2.

The basic importance of composition operators is evident already in the following observation. Here algebras are understood to be unital, with algebra homomorphisms preserving the unit by definition.

Proposition 1 *A linear map $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ is an algebra homomorphism if an only if A is a composition operator, i.e., for some $\varphi \in \mathcal{A}(\mathbb{D})$, $A = C_\varphi$.*

Proof. It is straightforward to check that, given $\varphi \in \mathcal{A}(\mathbb{D})$, the operator C_φ is an algebra homomorphism: C_φ is linear; C_φ fixes the constant function 1; and $C_\varphi(pq) = (C_\varphi p)(C_\varphi q)$ for every $p, q \in \mathbb{C}[z]$.

Conversely, suppose that a linear map $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ is an algebra homomorphism, and set $\varphi = Az$ (where z denotes the $p(z) = z$). The fact that A is a homomorphism means in particular that $A1 = 1$ and A respects multiplication. Hence $A(z^n) = \varphi^n$ for each $n \geq 1$, and for any polynomial

$$p(z) = \sum_{n=0}^{N} a_n z^n,$$

linearity of A yields that

$$(Ap)(z) = \sum_{n=0}^{N} a_n (\varphi(z))^n = p \circ \varphi(z),$$

proving $A = C_\varphi$. ∎

With respect to the general Pólya-Schur problem, the following observation is fundamental. It is constructive and gives sufficient conditions for an operator to be a solution.

Theorem 1 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be arbitrary sets. Choose polynomials $\psi \in \mathcal{P}(\Omega_2)$ and $\varphi \in \mathbb{C}[z]$ such that $\varphi(\Omega_2) \subset \Omega_1$. Then the product-composition operator $A = M_\psi C_\varphi : \mathbb{C}[z] \to \mathbb{C}[z]$ is linear and has the property that*

$$A(\mathcal{P}(\Omega_1)) \subset \mathcal{P}(\Omega_2) \cup \{0\}.$$

The proof of this result is immediate. What is astonishing is that for a broad class of Pólya-Schur problems—including some open problems cited in the literature, as well as problems to which hitherto only non-constructive characterizations were known—the operators described in Theorem 1 represent *all* solutions having rank greater than 1, as shown in Theorem 3. The necessity of the condition in these cases is far from trivial, however. The demonstration of necessity uses the Structure Theorem stated in the next section, the proof of which is detailed in Section 4.

3.2 The Structure Theorem

This paper's main technical engine is the following structure theorem. It asserts that a broad class of operators mapping stable polynomials to stable functions consists entirely of product-composition operators and rank 1 operators. The structure theorem implies far-reaching results for general Pólya-Schur problems, as well as characterizations of linear operators on Hardy space that preserve outer functions. Here $\mathcal{S}(\Omega)$ denotes the set of all Ω-stable functions in $\mathcal{A}(\mathbb{D})$.

Theorem 2 (Structure Theorem) *Let $\Omega \subset \mathbb{D}$ be a non-empty connected open set. If a linear map $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ has the property that*

$$A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\},$$

then either:

1. *there exist a function $\psi \in \mathcal{S}(\Omega)$ and a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ such that $A(f) = \nu(f)\psi$, for all $f \in \mathbb{C}[z]$; or*
2. *there exist a function $\psi \in \mathcal{S}(\Omega)$ and a non-constant function $\varphi \in \mathcal{A}(\mathbb{D})$, where $\varphi(\Omega) \subset \mathbb{D}$, such that $A = M_\psi C_\varphi$.*

Corollary 2 (Continuous Extension Property) *Let $\Omega \subset \mathbb{D}$ be a non-empty connected open set. If a linear map $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ has the property that*

$$A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\},$$

then A either has rank 1 or it extends to a continuous injection

$$\widetilde{A} : H^2 \to \mathcal{A}(\Omega).$$

Proof. Under the hypothesis of the corollary, if A has rank at least 2, then case 2 of Theorem 2 holds, and $A = M_\psi C_\varphi$ is a product-composition operator, with $\psi \in \mathcal{S}(\Omega)$ and non-constant $\varphi \in \mathcal{A}(\mathbb{D})$ such that $\varphi(\Omega) \subset \mathbb{D}$. The composition operator C_φ extends in the obvious way to H^2. To see that

$$C_\varphi : H^2 \to \mathcal{A}(\Omega)$$

is continuous, observe that $f_n \to f$ in the H^2 norm, then $f_n \to f$ in $\mathcal{A}(\mathbb{D})$, in the topology of uniform convergence on compact sets, which is coarser than the H^2 topology. In particular, $f_n \to f$ uniformly on the compact set $\varphi(K) \subset \mathbb{D}$, for any compact set $K \subset \Omega$. Thus $C_\varphi f_n \to C_\varphi f$ with

respect to the topology of uniform convergence on compact subsets of $\mathcal{A}(\Omega)$, establishing continuity of $C_\varphi : H^2 \to \mathcal{A}(\Omega)$. Since φ is non-constant and Ω is open, given $f, g \in H^2$, the equation

$$(C_\varphi f)(z) = (C_\varphi g)(z) \text{ for all } z \in \Omega$$

implies that $f = g$, whereby $C_\varphi : H^2 \to \mathcal{A}(\Omega)$ is injective.

Note that $\mathcal{A}(\Omega) \supset \mathcal{A}(\mathbb{D})$ is an algebra, so that

$$M_\psi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$$

is a well defined mapping; that M_ψ is continuous with respect to the topology of uniform convergence on compact sets follows from boundedness of ψ on compact sets. Since ψ is Ω-stable, $M_\psi : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is injective. Combining the above observations yields that

$$\widetilde{A} = M_\psi C_\varphi : H^2 \to \mathcal{A}(\Omega)$$

is a continuous injection. ∎

Corollary 3 (Algebra Embedding Property) *Let $\Omega \subset \mathbb{D}$ be a non-empty connected open set, and let*

$$A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$$

be a linear map that fixes the constant function 1*. If A has the property that*

$$A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\},$$

then either A has rank 1 or A is an embedding of the algebra $\mathbb{C}[z]$ into $\mathcal{A}(\mathbb{D})$.

Proof. Under the given hypothesis, if A has rank at least two then $A = M_\psi C_\varphi$ conforms to part 2 of Theorem 2. Moreover $\psi = A1 = 1$, so that $A = C_\varphi$ is a composition operator, with $\varphi \in \mathcal{A}(\mathbb{D})$ being non-constant. The latter fact renders C_φ injective, as in Corollary 2; and by Proposition 1, $A = C_\varphi$ is an algebra homomorphism. Thus A is an algebra embedding, as desired. ∎

Note that a homomorphism from $\mathbb{C}[z]$ to $\mathcal{A}(\mathbb{D})$ may have rank 1, but not every rank 1 operator that transforms stability is a homomorphism.

3.3 General Pólya-Schur Problems

Theorem 3 (General Pólya-Schur Solution) *Suppose $\Omega_1 \subset \mathbb{C}$ is bounded, and $\Omega_2 \subset \mathbb{C}$ has non-empty interior. A linear map $A : \mathbb{C}[z] \to \mathbb{C}[z]$ has the property that $A(\mathcal{P}(\Omega_1)) \subset \mathcal{P}(\Omega_2) \cup \{0\}$ if and only if either:*

1. *there exist a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ and a polynomial $\psi \in \mathcal{P}(\Omega_2)$ such that $A(f) = \nu(f)\psi$, for all $f \in \mathbb{C}[z]$; or*
2. *there exist $\psi \in \mathcal{P}(\Omega_2)$ and a non-constant polynomial φ for which $\varphi(\Omega_2) \subset \Omega_1$ such that $A = M_\psi C_\varphi$.*

Proof. Given $\tau > 0$, let $D_\tau : \mathbb{C}[z] \to \mathbb{C}[z]$ denote the dilation operator, acting on each $p \in \mathbb{C}[z]$ as

$$(D_\tau p)(z) = p(\tau z), \text{ for every } z \in \mathbb{C}.$$

Note that for any $\Omega \subset \mathbb{C}$, $D_\tau(\mathcal{P}(\Omega)) = \mathcal{P}(\frac{1}{\tau}\Omega)$.

Now, let $A : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator such that

$$A(\mathcal{P}(\Omega_1)) \subset \mathcal{P}(\Omega_2) \cup \{0\},$$

where Ω_1 is bounded and Ω_2 has non-empty interior. Since Ω_1 is bounded, there exists a $\delta > 0$ such that $\delta\Omega_1 \subset \mathbb{D}$. Thus $\mathcal{P}(\mathbb{D}) \subset \mathcal{P}(\delta\Omega_1)$ and the operator D_δ maps $\mathcal{P}(\delta\Omega_1)$ into $\mathcal{P}(\Omega_1)$. So, by hypothesis,

$$AD_\delta(\mathcal{P}(\mathbb{D})) \subset \mathcal{P}(\Omega_2) \cup \{0\}.$$

Since Ω_2 has non-empty interior, there exist an $\varepsilon > 0$ and an open connected set $\Omega \subset \mathbb{C}$ such that

$$\Omega \subset \mathbb{D} \cap \varepsilon\Omega_2.$$

Since $D_{1/\varepsilon}$ maps $\mathcal{P}(\Omega_2)$ into $\mathcal{P}(\varepsilon\Omega_2)$ and, by choice of Ω,

$$\mathcal{P}(\varepsilon\Omega_2) \subset \mathcal{P}(\Omega) \subset \mathcal{S}(\Omega),$$

it follows that

$$D_{1/\varepsilon}AD_\delta(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\}.$$

Thus the structure of the operator $\widetilde{A} = D_{1/\varepsilon}AD_\delta$ is given by Theorem 2. In detail, either:

1. there exist a function $\widetilde{\psi} \in \mathcal{S}(\Omega)$ and a linear functional $\widetilde{\nu} : \mathbb{C}[z] \to \mathbb{C}$ such that $\widetilde{A}(f) = \widetilde{\nu}(f)\widetilde{\psi}$, for all $f \in \mathbb{C}[z]$; or

2. there exist a function $\widetilde{\psi} \in \mathcal{S}(\Omega)$ and a non-constant function $\widetilde{\varphi} \in \mathcal{A}(\mathbb{D})$, where $\widetilde{\varphi}(\Omega) \subset \mathbb{D}$, such that $\widetilde{A} = M_{\widetilde{\psi}} C_{\widetilde{\varphi}}$.

In the former case,

$$A(f) = \left(\widetilde{\nu} D_{1/\delta}(f)\right) D_\varepsilon \widetilde{\psi} = \nu(f)\psi,$$

where $\psi = D_\varepsilon \widetilde{\psi}$ and $\nu = \widetilde{\nu} D_{1/\delta}$. And in the latter case,

$$A = D_\varepsilon M_{\widetilde{\psi}} C_{\widetilde{\varphi}} D_{1/\delta} = M_\psi C_\varphi,$$

where $\psi = D_\varepsilon \widetilde{\psi}$ and $\varphi = \frac{1}{\delta} D_\varepsilon \widetilde{\varphi}$.

Thus far it has been proven that if $A : \mathbb{C}[z] \to \mathbb{C}[z]$ is a linear operator such that

$$A(\mathcal{P}(\Omega_1)) \subset \mathcal{P}(\Omega_2) \cup \{0\},$$

then A is either a rank 1 operator mapping $f \mapsto \nu(f)\psi$ or a product-composition operator of the form $M_\psi C_\varphi$. It still remains to prove that $\psi \in \mathcal{P}(\Omega_2)$ and that the non-constant function φ is a polynomial such that $\varphi(\Omega_2) \subset \Omega_1$; this is rather straightforward. If A is the zero operator then there is nothing to prove, since ν can be taken to be the zero functional, and ψ can be taken arbitrarily; in particular one may take $\psi \in \mathcal{P}(\Omega_2)$. If A is non-zero of rank 1, then, by hypothesis, $A1 = \nu(1)\psi \in \mathcal{P}(\Omega_2)$, since $1 \in \mathcal{P}(\Omega_1)$, and hence $\psi \in \mathcal{P}(\Omega_2)$. Otherwise $A = M_\psi C_\varphi$ has rank at least 2, and $A1 = \psi \in \mathcal{P}(\Omega_2)$. To see that φ is a polynomial such that

$$\varphi(\Omega_2) \subset \Omega_1,$$

let $z_0 \in \mathbb{C} \setminus \Omega_1$, so that $p(z) = z - z_0$ belongs to $\mathcal{P}(\Omega_1)$. Then $Ap \in \mathcal{P}(\Omega_2) \cup \{0\}$, where

$$(Ap)(z) = \psi(z)\varphi(z) - z_0\psi(z).$$

Note that because A has rank at least 2, φ is not constant, and so Ap cannot be 0. Therefore,

$$Ap = \psi(\varphi - z_0) \in \mathcal{P}(\Omega_2),$$

which—given that $\psi \in \mathcal{P}(\Omega_2)$—implies that for all $z \in \Omega_2$, $\varphi(z) \neq z_0$. Since $z_0 \in \mathbb{C} \setminus \Omega_1$ was arbitrary, it follows that for every $z \in \Omega_2$, $\varphi(z) \in \Omega_1$, as desired. That $\varphi \in \mathcal{A}(\mathbb{D})$ is actually a polynomial follows from the fact that $Az^n = \psi\varphi^n$ is a polynomial for each $n \geq 0$.

The converse direction of the theorem is much simpler. If $A = M_\psi C_\varphi$ where $\psi \in \mathcal{P}(\Omega_2)$ and φ is a polynomial such that $\varphi(\Omega_2) \subset \Omega_1$, then it follows immediately that

$$A(\mathcal{P}(\Omega_1)) \subset \mathcal{P}(\Omega_2) \cup \{0\},$$

and similarly if $A(f) = \nu(f)\psi$ for every $f \in \mathbb{C}[z]$, where $\nu : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional. ∎

3.4 Outer Preserving Operators

Let $\mathcal{O}$ denote the set of all outer functions in H^2, and let $\widetilde{\mathcal{O}}$ denote the set of all shifted outer functions.

Lemma 4 *A bounded linear functional $\nu : H^2 \to \mathbb{C}$ satisfies*

$$\nu(\mathcal{O}) \subset \mathbb{C} \setminus \{0\} \quad (\textit{respectively, } \nu(\widetilde{\mathcal{O}}) \subset \mathbb{C} \setminus \{0\})$$

if and only if there exist a point $z_0 \in \mathbb{D}$ (respectively, $z_0 \in \mathbb{D} \setminus \{0\}$) and a scalar $\sigma \in \mathbb{C} \setminus \{0\}$ such that for all $f \in H^2$,

$$\nu(f) = \sigma f(z_0).$$

Proof. If there exist a point $z_0 \in \mathbb{D}$ and a scalar $\sigma \in \mathbb{C} \setminus \{0\}$ such that for all $f \in H^2$,

$$\nu(f) = \sigma f(z_0),$$

then $\nu : H^2 \to \mathbb{C}$ is bounded (as evaluation at a point in $\mathbb{D}$ is a bounded linear functional on H^2) and if $f \in \mathcal{O}$ then $\sigma f(z_0) \neq 0$, since outer functions have no zeros in the interior of the unit disk. Similarly for the case of shifted outer functions $\widetilde{\mathcal{O}}$ with $z_0 \in \mathbb{D} \setminus \{0\}$.

For the converse implication, suppose that $\nu : H^2 \to \mathbb{C}$ is a bounded linear functional such that

$$\nu(\mathcal{O}) \subset \mathbb{C} \setminus \{0\}.$$

Set $\rho_n = \nu(z^n)$ for each $n \geq 0$; and for each $\xi \in \mathbb{D}$, let $g_\xi \in H^2$ denote the function

$$g_\xi(z) = \sum_{n=0}^{\infty} (\xi z)^n.$$

Since ν is well-defined and bounded, the series

$$\nu(g_\xi) = \sum_{n=0}^{\infty} \rho_n \xi^n$$

converges for every $\xi \in \mathbb{D}$. In particular, the inequality $|\rho_n| < r^{-n}$ holds eventually for each fixed $0 < r < 1$. For each $w \in \mathbb{C}$, let f_w denote the scaled exponential

$$f_w(z) = e^{wz}.$$

Each $f_w \in \mathcal{O}$, and therefore by hypothesis the function $F(w)$ defined as

$$F(w) = \nu(f_w) = \sum_{n=0}^{\infty} \rho_n \frac{w^n}{n!}$$

is zero-free. Moreover, the eventual inequality $|\rho_n| < r^{-n}$ shows F to be entire of order at most 1, whence

$$F(w) = e^{\alpha+\beta w},$$

for some scalars α and β, by Hadamard's Theorem. It follows that

$$\rho_n = e^{\alpha}\beta^n,$$

for each $n \geq 0$. Furthermore, the Riesz representation theorem for linear functionals implies that there exists a $g \in H^2$ such that

$$\nu(f) = \langle f, g\rangle \text{ for every } f \in H^2.$$

The coefficients of $g(z) = \sum_{n=0}^{\infty} a_n z^n$ are given by

$$\overline{a}_n = \nu(z^n) = \rho_n = e^{\alpha}\beta^n;$$

therefore g has the power series expansion

$$g(z) = e^{\alpha}\sum_{n=0}^{\infty} \overline{\beta}^n z^n,$$

and $|\beta| < 1$ since $g \in H^2$. Thus $\langle f, g\rangle = e^{\alpha} f(\beta)$. Setting $z_0 = \beta$ and $\sigma = e^{\alpha}$, this shows that

$$\nu(f) = \sigma f(z_0),$$

as desired. The case where $\nu(\widetilde{\mathcal{O}}) \subset \mathbb{C} \setminus \{0\}$ is similar, except that the fact that $z \in \widetilde{\mathcal{O}}$ implies that $\nu(z) = e^{\alpha}\beta \neq 0$, whereby $\beta \in \mathbb{D} \setminus \{0\}$. ∎

Theorem 4 *Let $A : H^2 \to H^2$ be a bounded linear operator such that*

$$A(\mathcal{O}) \subset \mathcal{O}.$$

Then there exist an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ and a function $\psi \in \mathcal{O}$ such that

$$A = M_{\psi} C_{\varphi}.$$

Proof. Since $\mathbb{C}[z] \subset H^2 \subset \mathcal{A}(\mathbb{D})$ and $\mathcal{P}(\mathbb{D}) \subset \mathcal{O} \subset \mathcal{S}(\mathbb{D})$, the hypothesis of the theorem implies that the restriction of A to polynomials is a linear operator of the form

$$A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$$

such that $A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\mathbb{D}) \cup \{0\}$. Thus Theorem 2 applies with $\Omega = \mathbb{D}$, and hence either:

1. there exist a function $\widetilde{\psi} \in \mathcal{S}(\mathbb{D})$ and a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ such that $A(f) = \nu(f)\widetilde{\psi}$ for every $f \in \mathbb{C}[z]$; or
2. there exist a function $\psi \in \mathcal{S}(\mathbb{D})$ and a non-constant analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, such that $A = M_\psi C_\varphi$.

In case 1, the hypothesis that $\nu(\mathcal{O})\widetilde{\psi} \subset (\mathcal{O})$ implies that $\widetilde{\psi} \in \mathcal{O}$ and $\nu(\mathcal{O}) \subset \mathbb{C} \setminus \{0\}$. Since A is bounded, this implies in turn that $\nu : H^2 \to \mathbb{C}$ is a bounded linear functional, since $||\widetilde{\psi}\nu(f)|| = ||\widetilde{\psi}||\,|\nu(f)|$ for each $f \in H^2$. Therefore, by Lemma 4, the linear functional ν is proportional to evaluation at a point: for every $f \in H^2$,

$$\nu(f) = \sigma f(z_0), \text{ for some } z_0 \in \mathbb{D} \text{ and some non-zero } \sigma \in \mathbb{C}.$$

Thus, letting $\varphi : \mathbb{D} \to \mathbb{D}$ denote the constant function $\varphi(z) = z_0$, and setting $\psi = \sigma\widetilde{\psi}$, the rank 1 operator A has the form $A = M_\psi C_\varphi$, in conformity with the conclusion of the present theorem.

In case 2, where A has the form $M_\psi C_\varphi$, the fact that $1 \in \mathcal{O}$ implies that $\psi = A1 \in \mathcal{O}$, giving the desired conclusion once again. ∎

The following fact about shifted outer functions, which is needed for the proof of Theorem 5, follows easily from the standard integral representation for outer functions.

Proposition 5 *Let $f, g \in H^2$. If $f \in \widetilde{\mathcal{O}}$ and $fg \in \widetilde{\mathcal{O}}$, then $g \in \widetilde{\mathcal{O}}$.*

Theorem 5 (See [8]) *Let $A : H^2 \to H^2$ be a bounded linear operator such that*

$$A\left(\widetilde{\mathcal{O}}\right) \subset \widetilde{\mathcal{O}}.$$

Then there exist functions $\psi, \varphi \in \widetilde{\mathcal{O}}$, where $\varphi : \mathbb{D} \to \mathbb{D}$, such that

$$A = M_\psi C_\varphi.$$

Proof. The proof is very similar to that of Theorem 4. Setting $\Omega = \mathbb{D}\setminus\{0\}$, note that $\mathcal{P}(\mathbb{D}) \subset \widetilde{\mathcal{O}} \subset \mathcal{S}(\Omega)$; thus the hypothesis of the theorem implies that

$$A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D}),$$

with $A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\}$. Theorem 2 therefore implies that either:

1. there exist a function $\widetilde{\psi} \in \mathcal{S}(\mathbb{D})$ and a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ such that $A(f) = \nu(f)\widetilde{\psi}$ for all $f \in \mathbb{C}[z]$; or

2. there exist a function $\psi \in \mathcal{S}(\mathbb{D})$ and a non-constant function $\varphi \in \mathcal{A}(\mathbb{D})$, where $\varphi(\Omega) \subset \mathbb{D}$, such that $A = M_\psi C_\varphi$.

In case 1, as in the proof of Theorem 4 above, Lemma 4 yields that the linear functional ν is proportional to evaluation at a point: for every $f \in H^2$,

$$\nu(f) = \sigma f(\zeta), \text{ for some } \zeta \in \Omega \text{ and some non-zero } \sigma \in \mathbb{C},$$

with the difference that $\zeta \neq 0$, as per the part of Lemma 4 pertaining to $\widetilde{\mathcal{O}}$. As before, letting $\varphi : \mathbb{D} \to \mathbb{D}$ denote the constant function $\varphi(z) = \zeta$, and setting $\psi = \sigma\widetilde{\psi}$, the rank 1 operator A has the form $A = M_\psi C_\varphi$, where the non-zero constant function φ belongs to $\widetilde{\mathcal{O}}$.

In case 2, where A has the form $M_\psi C_\varphi$, the fact that $1 \in \widetilde{\mathcal{O}}$ implies that $\psi = A1 \in \widetilde{\mathcal{O}}$. And the identity function $f(z) = z$ belongs to $\widetilde{\mathcal{O}}$, so $\psi\varphi = Af \in \widetilde{\mathcal{O}}$ also. By Proposition 5 this implies that $\varphi \in \widetilde{\mathcal{O}}$. Since $\varphi \in \mathcal{A}(\mathbb{D})$ is analytic, the property $\varphi(\Omega) \subset \mathbb{D}$ given by Theorem 2 implies further that $\varphi(\mathbb{D}) \subset \mathbb{D}$, completing the proof. ∎

4 Proof of the Structure Theorem

4.1 Characteristic functions

4.1.1 Moment functions

A linear map $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ is evidently determined by its action on monomials $\mu_n(z) = z^n$, for which notation is fixed as follows. For each $n \geq 0$, let

$$\psi_n = A\mu_n$$

denote the *nth moment of A*.

The proof of the main structure theorem requires the following hypothesis.

Let $\Omega \subset \mathbb{D}$ be a nonempty connected open set, and suppose that $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ is a linear map, different from the zero operator, such that (H)

$$A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\}.$$

The first step is to assess the implications of this hypothesis for the moments ψ_n of A. It turns out that either all the moments ψ_n lie on a single complex line, or else for each $z \in \Omega$, the numbers $|\psi_n(z)|$ are bounded uniformly in n. This dichotomy is expressed in detail by the following two propositions.

Proposition 6 *Suppose hypothesis (H) holds and that ψ_0 is identically zero. Then there exists $\varphi \in \mathcal{A}(\mathbb{D})$ such that $\psi_n \in \mathbb{C}\varphi$ for every $n \geq 0$; i.e., the operator A has rank 1.*

Proof. Since A is not the zero operator then for some $n \geq 1$, the nth moment $\varphi = \psi_n$ is not identically zero. Since φ is analytic and Ω contains a condensation point, there exists a point $\zeta \in \Omega$ at which $\varphi(\zeta) \neq 0$. Given an arbitrary ψ_m, there is a choice of $a \in \mathbb{C}$ such that

$$\psi_m(\zeta) - a\varphi(\zeta) = 0,$$

whereby the function $\psi_m - a\varphi$ is not Ω-stable. Note that for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq |a| + 1$, the polynomial

$$p(z) = \alpha + z^m - az^n$$

is $\mathbb{D}$-stable. Therefore by (H) its image $Ap = \psi_m - a\varphi$ is either Ω-stable or identically 0. The former possibility has been ruled out, forcing $\psi_m = a\varphi$. Thus each of the moments of A lies in the line $\mathbb{C}\varphi$. ∎

If all its moments are scalar multiples of a fixed function φ, then the operator A has rank 1 and has the form

$$Ap = \nu(p)\varphi,$$

where $\nu : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional. This represents the degenerate case of the main structure theorem. The following proposition addresses the non-degenerate case.

Proposition 7 *Suppose the hypothesis (H) holds and that A has rank at least 2. Then for each $n \geq 1$, the nth moment of A is subject to the following bounds.*

1. *If $\psi_n \notin \mathbb{C}\psi_0$ then $\left|\dfrac{\psi_n(z)}{\psi_0(z)}\right| < 1$ for every $z \in \Omega$.*

2. If $\psi_n \in \mathbb{C}\psi_0$ *then* $\left|\dfrac{\psi_n(z)}{\psi_0(z)}\right| < 3$ *for every* $z \in \mathbb{D}$.

Proof. Since A has rank greater than 1, it follows from Proposition 6 that ψ_0 is not identically 0. And ψ_0 is therefore Ω-stable by (H).

Suppose that $\psi_n \notin \mathbb{C}\psi_0$, for some fixed $n \geq 1$, and let $\zeta \in \Omega$ be arbitrary. Note that for $a \in \mathbb{C}\backslash\mathbb{D}$ the polynomial $p(z) = a + z^n$ is $\mathbb{D}$-stable. The hypothesis (H) therefore implies that $Ap = a\psi_0 + \psi_n$ is either Ω-stable or identically zero. But $\psi_n \notin \mathbb{C}\psi_0$, so the latter possibility is ruled out. Since the value $a = -\psi_n(\zeta)/\psi_0(\zeta)$ renders p unstable, it follows that $-\psi_n(\zeta)/\psi_0(\zeta) \in \mathbb{D}$ and hence that $|\psi_n(\zeta)| < |\psi_0(\zeta)|$, proving part 1.

Next suppose that $\psi_n = \alpha\psi_0$ for some $\alpha \in \mathbb{C}$. By Proposition 6, there exists an $m \geq 1$ for which $\psi_m \notin \mathbb{C}\psi_0$, since A has rank at least two. Let $\zeta \in \Omega$, and set $a = -\psi_m(\zeta)/\psi_0(\zeta)$. Then $|a| < 1$, by part 1 above, and so the polynomial

$$p(z) = (a - \alpha) + z^m + z^n$$

is $\mathbb{D}$-stable as long as $|\alpha| \geq 3$. But

$$Ap = a\psi_0 - \alpha\psi_0 + \psi_m + \psi_n = a\psi_0 + \psi_m,$$

which has a zero at $\zeta \in \Omega$ and is not identically zero. Therefore p cannot be stable, forcing $|\alpha| < 3$. This proves part 2. ∎

4.1.2 Characteristic functions

In order to avoid repeatedly stating the rank two hypothesis, recall the notation $\mathcal{T}_2(\Omega)$, which denotes the set of all linear operators $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ such that

$$A \text{ has rank at least 2, and } A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega) \cup \{0\}.$$

Let $A \in \mathcal{T}_2(\Omega)$. Then for each $k \geq 1$, and each $(z, w) \in \Omega \times \mathbb{C}$, set

$$F_k(z, w) = \sum_{n=0}^{\infty} \psi_{kn}(z) \frac{w^n}{n!}. \tag{1}$$

Proposition 7 ensures that the coefficients $\psi_{kn}(z)$ are bounded in magnitude by $3|\psi_0(z)|$, for each $z \in \Omega$. The series on the right-hand side of (1) converges absolutely for every $(z, w) \in \Omega \times \mathbb{C}$, and hence that

$$F_k : \Omega \times \mathbb{C} \to \mathbb{C}$$

is a well-defined function determined by the operator A. Moreover, for each fixed $z \in \Omega$, $F_k(z, w)$ is an entire function of the variable w.

Definition 1 *Given an operator $A \in \mathcal{T}_2(\Omega)$, and an integer $k \geq 1$, the function F_k defined according to (1) is termed the k**th characteristic function of** A.*

4.1.3 Independence of zeros on $z \in \Omega$

Recall the order s of an entire function is defined as

$$s = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r},$$

where $M(r)$ is the maxmimum modulus of the entire function on the disk of radius r centred at the origin.

Proposition 8 *Let $A \in \mathcal{T}_2(\Omega)$. Then the following statements hold for each $k \geq 1$.*

1. *The characteristic function $F_k(z, w)$ is an entire function in w of order at most 1, and is analytic in $z \in \Omega$.*

2. *If $F_k(z_0, w_0) = 0$ for some $(z_0, w_0) \in \Omega \times \mathbb{C}$, then $F_k(z, w_0) = 0$ for all $z \in \Omega$.*

Proof. For fixed $z \in \Omega$, the function $F_k(z, w)$ is entire in w, by the uniform bound on $|\psi_{kn}(z)|$. That it has order at most 1 follows from the estimate

$$|F_k(z, w)| \leq 3|\psi_0(z)|e^{|w|}.$$

For each compact subset $K \subset \Omega$, the function ψ_0 is bounded on K whence the functions ψ_n are uniformly bounded on K by Proposition 7. So for fixed w, the series (1) is uniformly Cauchy on K, and thus the series converges to an analytic function in z, proving part 1.

Fix $w_0 \in \mathbb{C}$ and set

$$\sigma_n(z) = \sum_{m=0}^{n} \frac{(w_0 z^k)^m}{m!},$$

the nth partial sum of the Taylor series expansion of $e^{w_0 z^k}$. As $e^{w_0 z^k}$ is bounded away from zero on the closed disk, and the partial sums converge uniformly on this disk, the σ_n are eventually $\mathbb{D}$-stable. The images $A(\sigma_n)$ are either Ω-stable, or identically zero, and converge uniformly to $F_k(z, w_0)$. If $F_k(z, w_0)$ is not identically zero, it can be written as a limit of a subsequence

of Ω-stable functions $A(\sigma_n)$, and thus by Hurwitz's theorem, $F_k(z, w_0)$ has no zeros at all in Ω. ∎

Proposition 8 establishes that the collection of zeros

$$E_k(z) = \{w \in \mathbb{C} \mid F_k(z, w) = 0\}$$

is the same for all $z \in \Omega$, and so may be denoted without ambiguity as E_k. The order of each zero is also independent of z, as can be seen by the following argument. Choose a circular path γ in $\mathbb{C}$ centred at w_0 of sufficiently small radius that every other member of E_k lies strictly outside γ. Since $F_k(z, w)$ is analytic in z, the order of the zero w_0 is given by the integer valued function

$$\xi(z) = \frac{1}{2\pi i} \int_\gamma \frac{\frac{\partial F_k}{\partial w}(z, w)}{F_k(z, w)} \, dw,$$

which is continuous, and therefore constant. This observation, together with Proposition 8, proves the following.

Proposition 9 *Let $A \in \mathcal{T}_2(\Omega)$. Then for each $k \geq 1$ there is a sequence $\mathcal{E}_k$ and an integer $\nu_k \geq 0$ with the property that: for every fixed $z \in \Omega$, $\mathcal{E}_k$ is a list of the zeros of $F(z, w)$ (as a function in w) excluding $w = 0$, with each zero being repeated according to its multiplicity; ν_k is the order of* 0 *as a zero of $F_k(z, w)$.*

Of course the sequence $\mathcal{E}_k$ may be empty, finite or countable, depending on the operator A, but it is does not depend on z.

4.2 First and second companion functions

4.2.1 The product form of F_1

Proposition 10 *Let $A \in \mathcal{T}_2(\Omega)$. Then the characteristic function $F_1 : \Omega \times \mathbb{C} \to \mathbb{C}$ has the form*

$$F_1(z, w) = q(w) e^{\alpha(z) + \beta(z) w} \quad \textit{with} \quad q(w) = \prod_{w_n \in \mathcal{E}_1} e^{\frac{w}{w_n}} \left(1 - \frac{w}{w_n}\right),$$

where $e^{\alpha(z)} = \psi_0(z)$ for all $z \in \Omega$, and $q(w)$ is entire.

Note that the product over $\mathcal{E}_1$ is just the constant function 1 in the special case where $\mathcal{E}_1$ is the empty set, in which case $q(w) = w^{\nu_1}$.

Proof. By of Proposition 8, $F_1(z,w)$ is entire of order at most 1 as a function in w. Hadamard's Theorem implies that the genus of $F_1(z,w)$ as a function in w is either 0 or 1, and moreover that F_1 has respective Weierstrass product representation of the form either

$$F_1(z,w) \;=\; e^{\alpha(z)}\; w^{\nu_1} \prod_{w_n \in \mathcal{E}_1} \left(1 - \frac{w}{w_n}\right) \tag{2}$$

$$\text{or}\quad F_1(z,w) \;=\; e^{\alpha(z)+\beta(z)w}\; w^{\nu_1} \prod_{w_n \in \mathcal{E}_1} e^{\frac{w}{w_n}} \left(1 - \frac{w}{w_n}\right). \tag{3}$$

Proposition 9 guarantees that the zeros w_n (including their multiplicities) and the index ν_1 do not depend on z. The values α and β appearing in the Weierstrass product may depend on z.

Comparing the Taylor series expansion of (2) to the definition (1) of the characteristic function F_1 shows the moments of A all to be multiples of $e^{\alpha(z)}$, whereby A has rank 1; therefore by the assumption that $A \in \mathcal{T}_2(\Omega)$, $F_1(z,w)$ has the form (3). If $\beta(z)$ is constant, than A is again seen to have rank 1, so $\beta(z)$ is non-constant.

In fact the index $\nu_1 = 0$; to see this, note that, by definition, $F_1(z,0) = \psi_0$. Since A has rank at least 2, it follows from Proposition 6 that ψ_0 is not identically 0 and is hence Ω-stable. On the other hand, according to the formula (3), $F_1(z,0) = q(0)e^{\alpha(z)}$ which is Ω-stable only if $q(0) \neq 0$ whereby $\nu_1 = 0$. The given formula for q then yields that $q(0) = 1$, which proves that $\psi_0 = e^{\alpha}$. Hadamard's Theorem itself guarantees that $q(w)$ is entire. ∎

The function $q(w)$ is of sufficient importance to warrant a name, as follows.

Definition 2 *The **first companion function** associated to an operator $A \in \mathcal{T}_2(\Omega)$ is the entire function*

$$q(w) = e^{-\alpha(z)-\beta(z)w} F_1(z,w),$$

appearing in Proposition 10.

Proposition 10 makes it possible to express the moments ψ_n of A in terms of the Taylor coefficients c_n of the first companion function. It is evident (as already mentioned in the proof of Proposition 10) that $c_0 = q(0) = 1$. Note further that $c_1 = 0$; this follows from the pattern of the first few terms in the Taylor expansion of

$$e^{\frac{w}{w_n}} \left(1 - \tfrac{w}{w_n}\right) = 1 - \tfrac{1}{2}\left(\tfrac{w}{w_n}\right)^2 + \text{ higher order terms}.$$

Any product of such series is free of first order terms in w.

4.2.2 Formulas for the moments

Proposition 11 *Suppose $A \in \mathcal{T}_2(\Omega)$. Then, with respect to the representation for F_1 in Proposition 10, the moments ψ_n of A are given by the formula*

$$\psi_n(z) = e^{\alpha(z)} n! \sum_{j=0}^{n} c_{n-j} \frac{(\beta(z))^j}{j!} \quad \textit{for all } z \in \Omega.$$

Proof. Expanding $q(w)e^{\beta(z)w}$ as a power series in w yields that, for $z \in \Omega$,

$$\begin{aligned} F_1(z,w) &= q(w)e^{\alpha(z)+\beta(z)w} \\ &= e^{\alpha(z)} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} c_{n-j} \frac{(\beta(z))^j}{j!} \right) w^n. \end{aligned}$$

Comparison to the series (1) defining F_1 yields the desired representation of the moments. ∎

Knowing that $c_0 = 1$ and $c_1 = 0$, as determined at the end of Section 4.2.1, leads to the following.

Proposition 12 *Referring to Propostion 10, the function $\beta : \Omega \to \mathbb{C}$ has the following properties:*

1. *β is analytic, with $|\beta(z)| < 1$ for every $z \in \Omega$;*
2. *β has a meromorphic extension $\tilde{\beta} = \psi_1/\psi_0 : \mathbb{D} \to \mathbb{C}$.*

Proof. By Proposition 11, the equation

$$\psi_1/\psi_0 = (c_1 + c_0\beta)/c_0 = \beta$$

holds on Ω. Since the moments ψ_n of A are analytic (on $\mathbb{D}$) and ψ_0 is Ω-stable, it follows that $\beta : \Omega \to \mathbb{C}$ is analytic. Also, β is not constant, by Proposition 10, and so ψ_1 and ψ_0 are not proportional. Proposition 7 thus implies that $|\psi_1/\psi_0| < 1$ on Ω. The ratio ψ_1/ψ_0 is evidently a meromorphic extension of β to the open disk $\mathbb{D}$. ∎

Substitution of the formula from Proposition 11 into the series (1) also yields formulas for the higher characteristic functions F_k:

$$F_k(z,w) = e^{\alpha(z)} \sum_{n=0}^{\infty} \left((nk)! \sum_{j=0}^{nk} c_{nk-j} \frac{(\beta(z))^j}{j!} \right) \frac{w^n}{n!} \quad \text{for all } z \in \Omega. \tag{4}$$

4.2.3 Canonical products of genus 1

The present section summarizes some needed facts relating the rate of growth of an entire function having genus at most 1 to the order of its Taylor coefficients; results not proven here can be found in [9, Chapter 14] and in [12, Chapter 1].

To repeat the definition given in Section 4.1.3, the *order* s of an entire function $g(w)$ is defined as follows. For $r > 0$, let $M(r)$ denote the maximum value of $|g(w)|$ on the disk $|w| \leq r$. Then

$$s = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

The *type* t of $g(w)$ is then defined to be

$$t = \limsup_{r\to\infty} r^{-s} \log M(r).$$

Let $f(w)$ be an entire function of order at most 1, having at least one zero, such that $f(0) \neq 0$. Suppose further that $f(w)$ has finite type in the case where the order $s = 1$. Let $\{w_n\}_{n=0}^{N}$ be the sequence of zeros of f, in increasing order

$$|w_n| \leq |w_{n+1}|,$$

with each zero repeated according to its multiplicity. Either $N = \infty$ or N is a finite integer. In order to avoid having to make this distinction, the following formalism will be used: in the case where N is finite, set $w_{N+k} = \infty$ for each $k \geq 1$, so that the full sequence $\{w_n\}_{n=1}^{\infty}$ is always defined. With this formalism, $f(w)$ has a Weierstrass product representation of the form

$$e^{\alpha+\beta w} \prod_{n=1}^{\infty} e^{\frac{w}{w_n}} \left(1 - \frac{w}{w_n}\right).$$

The purpose of the present section is to study the canonical product factor of this representation,

$$q(w) = \prod_{n=1}^{\infty} e^{\frac{w}{w_n}} \left(1 - \frac{w}{w_n}\right).$$

To this end, for each $n \geq 1$, set $a_n = 1/|w_n|$, and write $a = \{a_n\}_{n=1}^{\infty}$. Thus

$$a_n \geq a_{n+1}$$

for each $n \geq 1$, and the sequence a is eventually 0 in the case where $f(w)$ has a finite number of zeros. There are two basic cases to consider:

$$\textbf{Case A}, \sum_{n=1}^{\infty} a_n < \infty; \qquad \textbf{Case B,} \sum_{n=1}^{\infty} a_n = \infty.$$

Suppose Case B holds. Since $\sum_{n=1}^{\infty} a_n$ diverges, it follows [9, Theorem 14.1.5] that the order of $q(w)$ is at least 1; therefore the order of q is exactly 1, since the order of f does not exceed 1. Let τ denote the type of q. The type imposes bounds on the Taylor coefficients of

$$q(w) = \sum_{n=0}^{\infty} c_n w^n,$$

as follows (see [9, Theorem 14.1.2]):

$$\tau = \frac{1}{e} \limsup_{n \to \infty} n|c_n|^{\frac{1}{n}}.$$

The type of q is finite (since this is true of f), and so for any $\varepsilon > 0$, the inequality

$$|c_n| < (\tau + \varepsilon)^n \left(\frac{e}{n}\right)^n$$

holds for all sufficiently large n. By Stirling's formula and the fact that $\varepsilon > 0$ is arbitrary, it follows in turn that for all sufficiently large n,

$$|c_n| < \frac{(\tau + \varepsilon)^n}{n!}. \tag{5}$$

Next suppose that Case A holds. Since $\gamma = \sum_{n=1}^{\infty} a_n$ is finite, it follows that the number

$$\sigma = \sum_{n=1}^{\infty} \frac{1}{w_n}$$

is finite. And moreover, q may be factored as

$$q(w) = e^{\sigma w} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right),$$

where the product part converges to an entire function,

$$\tilde{q}(w) = \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right).$$

Let $\tilde{c}_n$ denote the coefficients in the power series expansion of $\tilde{q}$:

$$\tilde{q}(w) = \sum_{n=0}^{\infty} \tilde{c}_n w^n.$$

Note that the coefficients $\tilde{c}_n$ are symmetric functions in the numbers $\frac{1}{w_n}$, and are bounded by the corresponding symmetric functions in the a_n:

$$|\tilde{c}_n| \leq \sum_{k_1 < k_2 < \cdots < k_n} a_{k_1} a_{k_2} \cdots a_{k_n}.$$

In particular, $\tilde{c}_0 = 1$ and $|\tilde{c}_1| \leq \sum a_n$. Write

$$\gamma_N = a_1 + a_2 + \cdots + a_N \text{ for each } N \geq 1, \text{ with } \gamma = \sum_{n=1}^{\infty} a_n,$$

consider $n \geq 2$, and let $N \geq n$. Given a fixed value for $a_1 + a_2 + \cdots + a_N$, the finite sum

$$\sum_{k_1 < k_2 < \cdots < k_n \leq N} a_{k_1} a_{k_2} \cdots a_{k_n}$$

is maximized in the case where all the terms a_n have the same value $a_n = \gamma_N / N$. Therefore

$$\begin{aligned} \sum_{k_1 < k_2 < \cdots < k_n \leq N} a_{k_1} a_{k_2} \cdots a_{k_n} &\leq \binom{N}{n} \left(\frac{\gamma_N}{N}\right)^n \\ &= \frac{(\gamma_N)^n}{n!} \frac{N(N-1)\cdots(N-n+1)}{N^n} \\ &< \frac{(\gamma_N)^n}{n!} \\ &\leq \frac{\gamma^n}{n!}. \end{aligned}$$

Letting $N \to \infty$ yields that

$$|\tilde{c}_n| \leq \gamma^n / n!. \tag{6}$$

The formula $q(w) = e^{\sigma w} \tilde{q}(w)$ shows that the Taylor coefficients of q and $\tilde{q}$ are related by the equation

$$c_n = \sum_{j=0}^{n} \tilde{c}_{n-j} \frac{\sigma^j}{j!}$$

for each $n \geq 0$. The bound (5) therefore yields that

$$|c_n| \leq \sum_{j=0}^{n} \frac{\gamma^{n-j}}{(n-j)!} \frac{|\sigma|^j}{j!} = \frac{(\gamma + |\sigma|)^n}{n!}.$$

Together with the inequality (6), this proves the following.

Proposition 13 *Given a canonical product of the form*

$$q(w) = \prod_{n=1}^{\infty} e^{\frac{w}{w_n}} \left(1 - \frac{w}{w_n}\right) = \sum_{n=0}^{\infty} c_n w^n$$

whose type is at most 1 in the case where the order is exactly 1, there exists a $y > 0$ such that the Taylor coefficients satisfy

$$|c_n| = O\left(\frac{y^n}{n!}\right).$$

4.2.4 Zeros of the second companion function

Lemma 14 *Let $q(w) = \sum_{n=0}^{\infty} c_n w^n$ be the first companion function of a given operator $A \in \mathcal{T}_2(\Omega)$. Then there exists a $y > 0$ such that*

$$|c_n| = O\left(\frac{y^n}{n!}\right),$$

whence the sequence

$$\left\{(2n)! c_{2n} Y^{-n}\right\}_{n=0}^{\infty}$$

belongs to $l^2(\mathbb{Z}_+)$ for all for sufficiently large $Y > 0$.

Proof. This is a direct application of Proposition 13. The latter involves the notion of *type*, which is defined in Section 4.2.3. Before Proposition 13 can be applied, it is necessary to verify that q has order at most 1, and that in the case where q has order exactly 1, the type of q is finite. To that end, note that, as in the proof of Proposition 8, F_1 satisfies the bound

$$|F_1(z, w)| \leq 3|\psi_0(z)| e^{|w|},$$

for all $(z, w) \in \Omega \times \mathbb{C}$. And by Defintion 2,

$$q(w) = e^{-\alpha(z) - \beta(z) w} F_1(z, w) = \frac{1}{\psi_0(z)} e^{-\beta(z) w} F_1(z, w),$$

so that

$$|q(w)| \leq 3e^{|\beta(z)+1||w|}.$$

Applying the bound on β in part 1 of Propostion 12 produces the estimate $|q(w)| \leq 3e^{2|w|}$ for all $w \in \mathbb{C}$, from which it follows that q has order at most 1, and that the type is at most 2 in the case where q has order exactly 1. Thus q conforms to the conclusion of Proposition 13, and the lemma follows immediately. ∎

Definition 3 *The* ***second companion function*** *of an operator $A \in \mathcal{T}_2(\Omega)$ is the function $G : \Omega \times \mathbb{C} \to \mathbb{C}$ given by the formula*

$$G(z,w) = e^{-w(\beta(z))^2 - \alpha(z)} F_2(z,w),$$

where: F_2 is the second characteristic function of A; for every $z \in \Omega$, $e^{\alpha(z)} = \psi_0(z)$ and $\beta(z) = \psi_1(z)/\psi_0(z)$.

Since F_2 is entire in w of order at most 1, and analytic in z (see part 1 of Proposition 8), and since $e^{-w\beta(z)^2-\alpha(z)}$ has these same properties, it follows that the companion function $G(z,w)$ of an operator $A \in \mathcal{T}_2(\Omega)$ is entire in w of order at most 1, and analytic in z. Furthermore, for every $(z,w) \in \Omega \times \mathbb{C}$,

$$F_2(z,w) = 0 \text{ if and only if } G(z,w) = 0.$$

From the technical standpoint, a key step toward the proof of Theorem 6, is to show that G, and hence F_2, has no zeros. (The second companion function G turns out to be more revealing than F_2 itself.) Equation (4) makes it possible to express G in terms of the coefficients c_n of the first companion function $q(w)$. The following combinatorial identity helps to verify the resulting representation.

Lemma 15 *For any integer $n \geq 0$, sequence $\{c_k\}_{k=0}^{\infty}$ and variable β,*

$$\sum_{m=0}^{n} \sum_{j=0}^{m} c_{2m-j} \frac{(2m-j)!}{j!(m-j)!(n-m)!} 2^j \beta^{j+2n-2m} = \sum_{j=0}^{2n} c_{2n-j} \frac{(2n)!}{j!n!} \beta^j.$$

Proof. Using the multinomial coefficients defined as

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1!k_2!k_3!}, \text{ where } k_1 + k_2 + k_3 = n,$$

one expands the identity $(2 + z^2 + z^{-2})^n = (z + z^{-1})^{2n}$ to obtain

$$\sum_{k_1+k_2+k_3=n} \binom{n}{k_1, k_2, k_3} 2^{k_1} z^{2k_2} z^{-2k_3} = \sum_{j=0}^{2n} \binom{2n}{j} z^{-j} z^{2n-j}.$$

Change variables on the LHS to $j = k_1, m - j = k_2, n - m = k_3$ to obtain

$$\sum_{m=0}^{n}\sum_{j=0}^{m}\frac{n!}{j!(m-j)!(n-m)!}2^j z^{-2(j+n-2m)} = \sum_{j=0}^{2n}\frac{(2n)!}{j!(2n-j)!}z^{-2(j-n)}.$$

Now set $z^{-2k} = c_{n-k}\beta^{n+k}(n-k)!/n!$ for $k = -n, -n+1, \ldots, n$. Then on the LHS one has $z^{-2(j+n-2m)} = c_{2m-j}\beta^{j+2n-2m}(2m-j)!/n!$, and on the RHS, $z^{-2(j-n)} = c_{2n-j}\beta^j(2n-j)!/n!$. This substitution yields the desired identity. ∎

Proposition 16 *In terms of the functions $q(w) = \sum_{n=0}^{\infty} c_n w^n$ and $\beta(z)$ appearing in Proposition 10, the companion function G of the given operator A has the form*

$$\begin{aligned} G(z,w) &= \sum_{m,n=0}^{\infty} c_{m+2n}\frac{(m+2n)!}{m!n!}(2\beta(z))^m w^{m+n} & (7)\\ &= \sum_{m=0}^{\infty}\left((2w)^m\sum_{n=0}^{\infty}c_{m+2n}(m+2n)!\frac{w^n}{n!}\right)\frac{(\beta(z))^m}{m!} & (8)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}c_{2n-j}(2n-j)!\binom{n}{j}(2\beta(z))^j\right)\frac{w^n}{n!} & (9)\end{aligned}$$

Proof. Given Lemma 14, it is straightforward to check that the formulations (7), (8) and (9) are equivalent to one another; version (9) is convenient because it is a power series in w, which coincides with the form (4) of the characteristic functions. To prove (9), it suffices by Definition 3 to expand the expression

$$e^{\alpha(z)}e^{wz^2}G(z,w)$$

as a power series and show that it agrees with $F_2(z,w)$, i.e., the expression (4), with $k = 2$. Eliminating the factor $e^{\alpha(z)}$, one obtains

$$\begin{aligned} &G(z,w)e^{wz^2}\\ &= \left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}c_{2n-j}(2n-j)!\binom{n}{j}(2\beta(z))^j\right)\frac{w^n}{n!}\right)\cdot\left(\sum_{n=0}^{\infty}(\beta(z))^{2n}\frac{w^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{m!}\left(\sum_{j=0}^{m}c_{2m-j}(2m-j)!\binom{m}{j}(2\beta(z))^j\right)\frac{\beta^{2(n-m)}}{(n-m)!}\right)w^n\end{aligned}$$

and compares this to expansion of $e^{-\alpha}F_2$,

$$\sum_{n=0}^{\infty}\left(\frac{(2n)!}{n!}\sum_{j=0}^{2n} c_{2n-j}\frac{(\beta(z))^j}{j!}\right)w^n.$$

That the two series agree follows directly from Lemma 15. ∎

The given formulation for the second companion function G is well-suited to the analysis of zeros.

Proposition 17 *If $A \in \mathcal{T}_2(\Omega)$, then the second characteristic function F_2 has no zeros.*

Proof. Suppose $F_2(z_0, w_0) = 0$ for some $(z_0, w_0) \in \Omega \times \mathbb{C}$. By part 2 of Proposition 8, it follows that $F_2(z, w_0) = 0$ for all $z \in \Omega$, and hence that $G(z, w_0) = 0$ for all $z \in \Omega$. Note that $w_0 \neq 0$, because by definition $F_2(z, 0) = e^{\alpha(z)}$ is Ω-stable. Since the function $\beta : \Omega \to \mathbb{D}$ is analytic and non-constant, the coefficients of $G(z, w_0)$ expressed as a power series in β must all be zero. Using the formulation (8), this implies that for each $m \geq 0$,

$$\sum_{n=0}^{\infty} c_{m+2n}(m+2n)!\frac{(w_0)^n}{n!} = 0. \tag{10}$$

Let sequences $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ and $\omega = \{\omega_m\}_{n=0}^{\infty}$ be defined by the respective formulas

$$\sigma_n = (2n)!c_{2n}Y^{-n}, \quad \omega_n = \frac{(\overline{w_0}Y)^n}{n!},$$

where $Y > 0$ is sufficiently large that $\sigma \in l^2(\mathbb{Z}_+)$. Lemma 14 guarantees that such a Y exists. Letting R denote the right shift operator on sequences,

$$R(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots),$$

equation (10) is equivalent to the condition that for every $m \geq 0$,

$$\langle \sigma, Y^m R^m \omega\rangle = Y^m\langle \sigma, R^m\omega\rangle = 0 \tag{11}$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on the sequence space $l^2(\mathbb{Z}_+)$. At this point it is helpful to invoke some standard facts concerning the Hardy space H^2, specifically two characterizations of outer functions. A function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$ is *outer* if and only if it satisfies Jensen's formula,

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\, d\theta.$$

This is automatically true if, for instance, f is analytic and zero-free on the closed disk $\overline{\mathbb{D}}$. A second, equivalent, characterization that f be outer is that the linear span of the collection of right shifts of the coefficient sequence $a = \{a_n\}_{n=0}^{\infty}$,

$$\operatorname{span}\left\{R^j a \,|\, j \geq 0\right\},$$

is dense in $l^2(\mathbb{Z}_+)$ (or, in other words, that a is a cyclic vector of the shift operator R). Now, since $e^{Y\bar{w}_0 z}$ is analytic and non-zero for $z \in \overline{\mathbb{D}}$, it is an outer function. And therefore its sequence of Taylor coefficients $\omega \in l^2(\mathbb{Z}_+)$ is a cyclic vector of the right shift operator. The linear span of vectors $R^m\omega$ is thus dense in $l^2(\mathbb{Z}_+)$. The condition (11) therefore implies that σ is orthogonal to every member of $l^2(\mathbb{Z}_+)$ whereby it is identically 0. But $c_0 = 1$, so this is a contradiction. Therefore $F_2(z_0, w_0) \neq 0$ for every $(z_0, w_0) \in \Omega\times\mathbb{C}$. ∎

Lemma 18 *Let $A \in \mathcal{T}_2(\Omega)$. If the second companion function G has no zeros, then the first companion function $q = 1$ is constant.*

Proof. As usual, let α, β and $q(w) = \sum_{n=0}^{\infty} c_n w^n$ denote the functions appearing in Proposition 10 (so that $c_0 = 1$ and $c_1 = 0$). By Hadamard's Theorem and the comments following Defintion 3, if G has no zeros then it has the form

$$G(z,w) = e^{\gamma(z)+\epsilon(z)w} \text{ for every } (z,w) \in \Omega \times \mathbb{C},$$

for some functions $\gamma, \epsilon : \Omega \to \mathbb{C}$. By any of the formulas in Proposition 16, $G(z, 0) = c_0 = 1$, so $\gamma(z) = 0$ and

$$G(z,w) = e^{\epsilon(z)w} = \sum_{n=0}^{\infty} \epsilon(z)^n \frac{w^n}{n!}.$$

Comparing the coefficient of w^1 above, to that in (9), yields

$$\epsilon(z) = \sum_{j=0}^{1} c_{2-j}(2-j)!\binom{2}{j}(2\beta(z))^j = 2c_2 + 4c_1\beta(z) = 2c_2,$$

the latter equality since $c_1 = 0$. Thus $\epsilon(z)$ is constant and $G(z,w)$ does not depend on z. The function $\beta(z)$ is not constant, however. Therefore, the coefficients of positive powers of $\beta(z)$ in (8) must be identically 0. In particular, the coefficient of $\beta(z)^1$ is identically 0,

$$2w \sum_{n=0}^{\infty} c_{2n+1}(2n+1)!\frac{w^n}{n!} = 0,$$

from which it follows that $c_{2n+1} = 0$ for all $n \geq 0$. The same reasoning applied to the coefficient of $\beta(z)^2$ implies that

$$(2w)^2 \sum_{n=0}^{\infty} c_{2n+2}(2n+2)! \frac{w^n}{n!} = 0,$$

whereby $c_{2n+2} = 0$ for all $n \geq 0$. This proves that $q(w) = c_0 = 1$. ∎

4.2.5 Zeros of the first characteristic function

Proposition 19 *If $A \in \mathcal{T}_2(\Omega)$ then the characteristic function F_1 has the form*

$$F_1(z, w) = e^{\alpha(z)+\beta(z)w},$$

where $e^{\alpha(z)} = \psi_0(z)$, and $\beta(z) = \psi_1(z)/\psi_0(z)$ for every $z \in \Omega$.

Proof. This follows immediate from Propositions 10, 17 and Lemma 18. ∎

The final step in the argument is to infer the structure of an arbitrary $A \in \mathcal{T}_2(\Omega)$.

4.3 The product-composition theorem

Theorem 6 *If $A \in \mathcal{T}_2(\Omega)$, then there exist analytic functions $\psi, \varphi : \mathbb{D} \to \mathbb{C}$ with φ non-constant, where $\varphi(\Omega) \subset \mathbb{D}$, such that for every $f \in \mathbb{C}[z]$,*

$$(Af)(z) = \psi(z)f(\varphi(z)) \text{ for every } z \in \mathbb{D}.$$

In other words, $A = M_\psi C_\varphi$ is a product-composition operator.

Proof. By Proposition 19, F_1 has the form

$$F_1(z, w) = e^{\alpha(z)} \sum_{n=0}^{\infty} (\beta(z))^n \frac{w^n}{n!},$$

while by definition,

$$F_1(z, w) = \sum_{n=0}^{\infty} \psi_n(z) \frac{w^n}{n!}.$$

Thus for each $n \geq 0$,

$$\psi_n(z) = e^{\alpha(z)}(\beta(z))^n \text{ for every } (z, w) \in \Omega \times \mathbb{C}.$$

Set $\varphi(z) = \psi_1(z)/\psi_0(z)$; then $\varphi : \mathbb{D} \to \mathbb{C}$ is a meromorphic extension of β such that $\varphi(\Omega) \subset \mathbb{D}$, as per Proposition 12, so that by the foregoing equation,

$$\psi_n(z) = \psi_0(z)(\varphi(z))^n \tag{12}$$

holds on Ω for every $n \geq 0$. Since Ω has a condensation point and the left- and right-hand sides of (12) are meromorphic on $\mathbb{D}$, the equation extends to all of $\mathbb{D}$. Moreover, since ψ_n is analytic on $\mathbb{D}$ for each n, it follows from (12) that φ has no poles on $\mathbb{D}$; for otherwise ψ_n itself would have a pole on $\mathbb{D}$ for all sufficiently large n. Thus $\varphi : \mathbb{D} \to \mathbb{C}$ is analytic, completing the proof. ▮

In general, if $A : \mathbb{C}[z] \to \mathcal{A}(\mathbb{D})$ has the property that $A(\mathcal{P}(\mathbb{D})) \subset \mathcal{S}(\Omega)$, then either A is the zero operator, A has rank 1, or $A \in \mathcal{T}_2(\Omega)$. If $A = 0$ or A has rank 1, then A acts according to the formula $A(f) = \nu(f)\psi$, where $\psi \in \mathcal{S}(\Omega)$ and $\nu : \mathbb{C}[z] \to \mathbb{C}$ is a linear functional, possibly the 0 functional. (For if a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ has the property that $\nu(\mathcal{P}(\mathbb{D})) = \{0\}$, then $\nu = 0$; this is because the span of $\mathcal{P}(\mathbb{D})$ is $\mathbb{C}[z]$. Hence if $\nu \neq 0$, then necessarily $\psi \in \mathcal{S}(\Omega)$.) Combining this observation with Theorem 6 gives the full Structure Theorem, as stated in Section 3.2.

5 Remarks

There are several points to be made concerning the scope of the main results presented in Section 3. Firstly, the implications of the results, even in cases where a characterization of stable-preserving operators is already known, are significant; a detailed example is worked out below in Section 5.1.1. Secondly, the assumption of boundedness on Ω_1 cannot be dropped. This is illustrated below in Section 5.1.2, where a broader sufficient condition is given for certain unbounded domains.

Concerning the Hardy space results, there is a point to be made about what is known as Schur stability and the potential pitfall of conflicting conventions for the z-transform: the geophysicists' version is the correct choice if stable polynomials are to be relevant. See Section 5.2.1. Also, the main structure theorem easily applies to other scenarios of interest in signal processing, beyond Theorems 4 and 5. See Section 5.2.3 below.

Finally, the generality of the hypothesis of Theorem 3 allows for a complete analysis of even very fanciful scenarios, such as where Ω_1 is the Koch snowflake and Ω_2 is an annulus; and it shows that in other, less fanciful scenarios, such as with $\Omega_1 = \mathbb{D}$ and Ω_2 the complement in $\mathbb{C}$ of a real interval $[a, b]$, only rank 1 operators can arise.

5.1 Bounded versus unbounded domains

There is a heuristic reason why the existing characterizations of stable preserving operators for circular domains are not constructive. Put simply, circular domains are not all equivalent; it matters whether or not a domain is bounded. This is because, viewing the class $\mathbb{C}[z]$ as a collection of mappings on the Riemann sphere, $\mathcal{R}$, every non-constant polynomial p is a continuous surjection $p : \mathcal{R} \to \mathcal{R}$ having $z = \infty$ as a fixed point. The point $z = \infty$ is the unique common fixed point of the non-constant elements of $\mathbb{C}[z]$, and consequently, for linear operators that preserve the set of Ω-stable polynomials there is an inherent difference between domains Ω with $\infty \in \overline{\Omega}$ (unbounded domains) and domains where $\infty \notin \overline{\Omega}$ (bounded domains). A uniform result valid for circular domains collectively is predisposed to obscure the inherent differences. In more concrete terms, differential operators arise in the context of unbounded domains but not that of bounded domains; see Section 5.1.2.

5.1.1 Non-constructive characterizations

The particular values $\Omega_1 = \Omega_2 = \mathbb{D}$ constitute a case of Problem A which conforms to the hypothesis of Theorem 3, and for which there is also a known characterization of the solution. The relevant result in [5] is Corollary 3; in the case of the unit disk $\mathbb{D}$ it states the following.

Theorem 7 (From Corollary 3 in [5]) *A linear map $A : \mathbb{C}[z] \to \mathbb{C}[z]$ satisfies $A(\mathcal{P}(\mathbb{D})) \subset \mathcal{P}(\mathbb{D}) \cup \{0\}$ if and only if either:*

1. *there exist a linear functional $\nu : \mathbb{C}[z] \to \mathbb{C}$ and a polynomial $\psi \in \mathcal{P}(\mathbb{D})$ such that $A(f) = \nu(f)\psi$ for every $f \in \mathbb{C}[z]$; or*

2. *letting $f_{w,n}(z) = (1 + wz)^n$, the polynomial $Af_{w,n}$ is $\mathbb{D}$-stable for every $w \in \mathbb{D}$ and $n \geq 0$.*

In other words, letting $\mathcal{M}$ denote the set of polynomials

$$\mathcal{M} = \{f_{w,n} \mid w \in \mathbb{D} \text{ and } n \geq 0\},$$

condition 2 of the characterization is exactly that $A(\mathcal{M}) \subset \mathcal{P}(\mathbb{D})$. That is to say, Theorem 7 reduces the case $\Omega_1 = \Omega_2 = \mathbb{D}$ of Problem A to:

Problem C Determine all linear operators $A : \mathbb{C}[z] \to \mathbb{C}[z]$ such that

$$A(\mathcal{M}) \subset \mathcal{P}(\mathbb{D}) \cup \{0\}.$$

Problem C itself was open prior to the present paper, and proving that solutions consist exclusively of product-composition operators is non-trivial. That the full solution set is given directly by Theorem 3, shows the power of the latter result even in cases where there is a known characterization.

5.1.2 Unbounded domains

The hypothesis that Ω_1 is bounded cannot be dropped from Theorem 3, because in the case of unbounded Ω_1 the class of stable-preserving operators can go beyond Theorem 1 to include product-composition-differential operators. Let $D : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})$ denote differentiation; for any entire function $\pi(w)$, the symbol $\pi(D)$ denotes a well-defined linear operator,

$$\pi(D) : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D}),$$

that preserves that class $\mathbb{C}[z]$ of polynomials. The following observation follows easily from Theorem 1 and the Guass-Lucas theorem, which states that for any polynomial p, the zeros of Dp lie in the convex hull of the zeros of p.

Theorem 8 *Let $\Omega \subset \mathbb{C}$ be such that its complement $\mathbb{C} \setminus \Omega$ is convex. If $\psi \in \mathcal{P}(\Omega)$, $\varphi \in \mathbb{C}[z]$ is such that $\varphi(\Omega) \subset \Omega$, and $\pi(w) = w^n$, then the operator*

$$A = M_\psi C_\varphi \pi(D) : \mathbb{C}[z] \to \mathbb{C}[z]$$

has the property that $A(\mathcal{P}(\Omega)) \subset \mathcal{P}(\Omega)$.

Many more examples arise in certain special cases. For instance, if $\Omega_1 = \Omega_2 = \mathbb{C} \setminus \mathbb{R}$, then the product-composition-differential operator

$$A = M_\psi C_\varphi \pi(D)$$

is a solution to Problem A whenever ψ has real roots, $\varphi(z) = az + b$, for some $a, b \in \mathbb{R}$, and $\pi(w)$ is the limit uniformly on compact sets of a sequence of functions having only real zeros.

5.2 Signal processing

The connection of Theorems 4 and 5 to digital signal processing is detailed in [8]. Essentially, the problem is to construct operators on digital signals that preserve the property of minimum phase, defined below.

5.2.1 Minimum phase

Letting $l^2(\mathbb{Z})$ serve as the space of all causal (meaning zero for all negative time) digital signals, a signal $a = (a_0, a_1, a_2, \ldots)$ is minimum phase if and only if the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is outer. The latter is called the geophysicists' z-transform of a, to distinguish it from the more common (engineers') z-transform, which is $f(1/z)$. With respect to signal processing, the two versions of the z-transform obviously capture the same information, and it is possible to work with either one. Given a finitely supported signal $a \in l^2(\mathbb{Z})$, a is minimum phase if and only if $f(z)$ is $\mathbb{D}$-stable, which is equivalent to $f(1/z)$ having all its poles and zeros inside the closed unit disk $\overline{\mathbb{D}}$. But whereas $f(z)$ is a polynomial, $f(1/z)$ is in general a non-polynomial rational function. Thus linear operators on $l^2(\mathbb{Z})$ that preserve finitely supported minimum phase signals correspond to operators $A : \mathbb{C}[z] \to \mathbb{C}[z]$ that preserve $\mathbb{D}$-stability. On the other hand, to work with $f(1/z)$ one has to consider $(\mathbb{C} \setminus \mathbb{D})$-stability for non-polynomial functions, and the connection to Pólya-Schur problems is less direct.

5.2.2 The question of sufficiency

Unlike the scenario for stable polynomials (Theorem 3), there is a delicate question that arises for product-composition operators

$$A = M_\psi C_\varphi : H^2 \to H^2$$

that preserve shifted outer (or just outer) functions. Namely, what are necessary and sufficient conditions on ψ and φ for the operator A to be a bounded operator on H^2? For instance, it is sufficient for ψ to be bounded on $\mathbb{D}$ and φ to be a shifted outer function that fixes that unit disk. But this pair of conditions is not necessary. See [8] for a more detailed discussion.

5.2.3 Alternate scenarios

Theorems 4 and 5 comprise the cases of greatest interest to applications in geophysical signal processing. But it should be clear that other possible scenarios are easily handled in a similar way by the application of Theorem 2. For instance one can consider operators $A : H^2 \to H^2$ that map outer functions either to outer functions or to 0. Then the solution is much the same as in Theorem 4, except that a slightly wider class of rank 1 operators

arises: operators where the linear functional $\nu : H^2 \to \mathbb{C}$ need not be evaluation at a point. Or one could consider operators on Hardy space that carry outer functions to shifted outer functions, etc.

5.3 Open problems

It would be of great interest to find a constructive solution to Problem A covering the cases where Ω_1 is unbounded. It seems as though the main difficulty consists in determining explicit conditions on the composition-differential part of a product-composition-differential operator in order that the latter should transform stability in the desired way.

6 Conclusion

The results presented in this paper comprise explicit, constructive solutions to a broad class of Pólya-Schur problems calling for a description of all linear operators on polynomials that transform stability between regions in the complex plane.

The description of the solutions in Theorem 3 as product-convolution operators built from polynomials of a specified type is novel in two ways. Firstly, the underlying regions in the plane for which the results hold are very general, with a comprehensive solution being given for operators that transform polynomials that are stable with respect to an arbitrary bounded region to those that are stable with respect to an arbitrary region having non-empty interior. Previously known results have been confined to regions whose boundary is a circle or a line; and results in the present paper cover some outstanding open cases of these. Secondly, the results given in the present paper are constructive, whereas the most general previously known characterizations for circular domains involve testing whether a certain characteristic function of the operator is in the complex Laguerre-Pólya class, which is non-constructive; see [5] and [3]. Thus the results derived in the present paper complement the substantial recent results of Borcea and Brändén.

A second contribution of the present paper is that it gives full, constructive solutions to Pólya-Schur type problems in the functional analytic setting of Hardy space. Indeed Theorems 4 and 5, are examples of the type of result concerning entire functions called for by Borcea and Brändén in [3, p. 567]. The Hardy space results were motivated by signal processing, with the specific requirement for constructive solutions, and consequently have numerous direct applications as outlined in [8].

Because the main structure theorem, Theorem 2, concerns linear operators into algebras of analytic functions rather than just polynomials, it applies simultaneously to the functional analytic setting of Hardy space and to the original Pólya-Schur setting of polynomials. This bridge between the two settings shows that polynomial problems have an inherent, albeit hidden, analytic character, as stated in Corollary 2.

Given that an explicit, constructive description is possible for the class of Pólya-Schur problems covered by Theorem 3, it is natural to ask whether a comparably explicit solution can be given for problems involving unbounded domains.

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7 References

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